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On the Problem of Sharp Exponents in Multivariate Nikolskii-Type Inequalities

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Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, w a weight-function, $p>0$, and $\alpha>0$. Assuming some regularity conditions on w and the boundary $\partial\Omega$ of Ω we prove that if grad $w \neq 0$ on the set Γ where w vanishes and if Γ is transversal to $\partial\Omega$ then there exists a positive constant C such that for any polynomial P of degree at most n we have $\|P\|_{p,\Omega} \leqslant Cn^{\alpha} \|P\,|w|^{\alpha}\|_{p,\Omega}$; furthermore the exponent α of n is optimal.

NOTATION

We denote by $\mathscr{P}_n$ the set of algebraic polynomials of a single or several variables (according to the context) of degree at most n. The set of 2π-periodic trigonometric polynomials of order at most n is denoted by H_n.

For $E \subset \mathbb{R}^N$, $\|\cdot\|_{p,E}$ is the usual norm $(1 \leqslant p \leqslant \infty)$ or pseudo-norm $(0<p<1)$ on E.

In this paper 2π-periodic functions are considered as functions defined on the (circle) $\mathbb{K} = \mathbb{R}/2\pi\mathbb{Z}$; $\mathbb{K}$ is provided with the metric

$$d_K(x, y) = \operatorname{Min}\{|x-y+2k|;\ k \in \mathbb{Z}\}.$$

For a 2π-periodic function f we define $\|f\|_p^* := \|f\|_{p,[0,2\pi]}$. The boundary of a set $\Omega \subset \mathbb{R}^N$ is denoted by $\partial\Omega$.

In every statement and proof we use a collection $C, C_1, C_2, \ldots,$ of positive constants. Obviously, these constants have not the same meaning in different occurences.

1. INTRODUCTION

In 1919 Schur [7] gave estimates that we can rewrite in the following form: let $p \in \mathscr{P}_n$ be a polynomial of a single variable and $I = [-1, 1]$; then

$$\|P\|_{\infty,I} \leqslant (n+1)\, \|xP\|_{\infty,I}. \tag{1}$$

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Furthermore using the classical Markov inequality, we get easily

$$\|P\|_{\infty, I} \leqslant (n+1)^2 \, \|(1-x)P\|_{\infty, I}. \tag{2}$$

More generally, let $\delta_1, \delta_2, \gamma_1, ..., \gamma_r$ be positive constants, $a_1, ..., a_r$ satisfying $|a_i| < 1$ $(i=1, ..., r)$, $a_i \neq a_j$ $(i \neq j)$, and $p > 0$, $\alpha > 0$. We set $w(x) = |1-x|^{\delta_1} \, |1+x|^{\delta_2} \, |x-a_1|^{\gamma_1} \cdots |x-a_r|^{\gamma_r}$ and $k = \operatorname{Max}\{2\delta_1, 2\delta_2, \gamma_1, ..., \gamma_r\}$. Then there exists a positive constant C such that for any $P \in \mathscr{P}_n$

$$\|P\|_{p,I} \leqslant C n^{k\alpha} \, \|P w^{\alpha}\|_{p,I}; \tag{3}$$

furthermore the exponent $k\alpha$ is sharp (see [1, 5]). If for example $w(x) = |x-a|$ then $k=1$ if $|a| < 1$ and $k=2$ if $|a| = 1$. We see that the exponent of n depends on the location of a in I (interior or boundary).

The problem of proving inequalities similar to (1), (2), and (3) (Nikolskii-type inequalities) in the N-dimensional case has been investigated in [2] in the particular case of the uniform norm and $\alpha = 1$ (the interval I is then replaced by a bounded open set $\Omega \subset \mathbb{R}^N$). Under the assumptions that $w \in C^s$ (s large) and that $\partial\Omega$ is C^2 in a neighborhood of any $a \in \partial\Omega \cap \{x;\ w(x) = 0\}$ it was shown that

$$\|P\|_{\infty,\Omega} \leqslant C n^d \, \|Pw\|_{\infty,\Omega},$$

where the optimal exponent d is effectively computable and depends only on the geometric relations between the boundary of Ω and the set where w vanishes.

An other related result was proved in [4]: let $\Omega \subset \mathbb{R}^N$; if

(i) Ω preserves Markov's inequality, that is, if for some positive constants C_1 and r and for any $p \in \mathscr{P}_n$

$$\|\partial P/\partial x_i\|_{p,\Omega} \leqslant C_1 n^r \, \|P\|_{p,\Omega} \qquad (i=1, ..., N),$$

(ii) for some positive constant d and for any $x \in \bar{\Omega}$ there exists $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leqslant d$ and $w^{(\alpha)}(x) \neq 0$,

then, for some $C_2 = C_2(\Omega, w)$ and for any $P \in \mathscr{P}_n$ we have

$$\|P\|_{p,\Omega} \leqslant C_2 n^{rd} \, \|Pw\|_{p,\Omega}.$$

Let $\bar{\Omega}$ be a compact subset of $\mathbb{R}^N$ whose boundary can be defined in a neighborhood of any $a \in \partial\Omega$ (using local coordinates) by $x_n = f(x_1, ..., x_{n-1})$, where f is a Lipschitz function. It is known (see [3]) that $\bar{\Omega}$ preserves the Markov inequality with exponent $r=2$; then if grad $w \neq 0$ (i.e., $d=1$) we have

$$\|P\|_{p,\Omega} \leqslant C n^2 \, \|Pw\|_{p,\Omega}.$$

The exponent 2 is optimal as shown by the following examples:

EXAMPLES. Let $N=2$, $\Omega=[-1,1]^2$, $w(x,y)=1-x$, and $P_n(x,y)=P_n^{(2,0)}(x)$ be the sequence of Jacobi polynomials (see [8, Chap. 4] for notation); then

$$C'n^2 \|P_n w\|_{p,\Omega} \leqslant \|P_n\|_{p,\Omega} \leqslant Cn^2 \|P_n w\|_{p,\Omega}.$$

An analogous example can be given with $\Omega=$ the unit disc with center at origin and $w(x,y)=1-y$.

We can remark in both examples proving sharpness of exponent 2 that the set $\Gamma=\{(x,y);\ w(x,y)=0\}$ is tangential to the boundary $\partial\Omega$ of Ω.

In order to obtain a smaller optimal exponent, additional assumptions are obviously needed. The previous examples suggest that Γ should not be tangential to $\partial\Omega$.

The purpose of this paper is to prove that if Γ is transversal to $\partial\Omega$ then the previous results can be improved: the optimal exponent of n is 1.

2. Statement of the Result

In order to make things clearer (and easier to write) we restrict the statement of the theorem and its proof to the 2-dimensional case. The result can be adapted to the N-dimensional case using heavier notation.

Assumptions. (i) $\Omega\subset\mathbb{R}^2$ is an open bounded set.

(ii) w is a C^1-function defined on a neighborhood of $\bar{\Omega}$ and such that $\Gamma=\{(x,y);\ w(x,y)=0\}$ is a regular curve; that is: for any $(x,y)\in\Gamma$, grad $w(x,y)\neq 0$.

(iii) The boundary $\partial\Omega$ of Ω is C^2 in a neighborhood of every $a\in\partial\Omega\cap\Gamma$.

(iv) Γ is transversal to $\partial\Omega$; that is, $\partial\Omega\cap\Gamma\neq\emptyset$ and for any $a\in\partial\Omega\cap\Gamma$, the tangent lines to $\partial\Omega$ and Γ at a are distinct.

The aim of this paper is to prove the following

THEOREM. Let $p>0$, $\alpha>0$. Under assumptions (i), (ii), (iii), and (iv), there exists a positive constant C such that, for any $P\in\mathscr{P}_n$,

$$\|P\|_{p,\Omega} \leqslant Cn^\alpha \|P\,|w|^\alpha\|_{p,\Omega}.$$

Furthermore, the exponent α of n is sharp.

3. A Particular Case

We first examine the case when Ω is the disc with radius R and center at origin and Γ cuts transversally $\partial\Omega$ at only two distinct points A and B.

3.1. *Some Lemmas*

Lemma 1. *Let* $\theta_1, \theta_2 \in \mathbb{K}$ *satisfying* $d = d_K(\theta_1, \theta_2) > 0$. *Then, for any* $\theta_0 \in \mathbb{K}$, *we have* $|\sin((\theta - \theta_1)/2) \sin((\theta - \theta_2)/2)| \geqslant (d/\pi^2)\,|\theta - \theta_0|$ for either $\theta \in [\theta_0 - d/4, \theta_0]$ or $\theta \in [\theta_0, \theta_0 + d/4]$.

Proof. We refer to Fig. 1 and we give the proof when $\theta_1 < \theta_2$, $d = \theta_2 - \theta_1$, and $\theta_0 \in [\theta_1, (\theta_1 + \theta_2)/2]$. (The proof is easily adapted to the other cases.)

We denote by G (H resp.) the midpoint of AB (BC resp.) and EF is a segment parallel to AB. We have length $(EI) = d/4$. The slope of AB is $s = (\sin^2(d/4))/(d/2)$ and since $d \leqslant \pi$,

$$s = \frac{\sin^2(d/4)}{(d/4)^2}(d/8) \geqslant \frac{\sin^2(\pi/4)}{(\pi/4)^2}(d/8) = d/\pi^2.$$

The segment EF whose slope is $\geqslant d/\pi^2$ lies under the graph of the function $x \to |\sin((\theta - \theta_1)/2) \sin((\theta - \theta_2)/2|$. Then

$$|\sin((\theta - \theta_1)/2) \sin((\theta - \theta_2)/2)|$$
$$\geqslant (d/\pi^2)\,|\theta - \theta_0| \qquad (\theta \in [\theta_0, \theta_0 + d/4]).$$

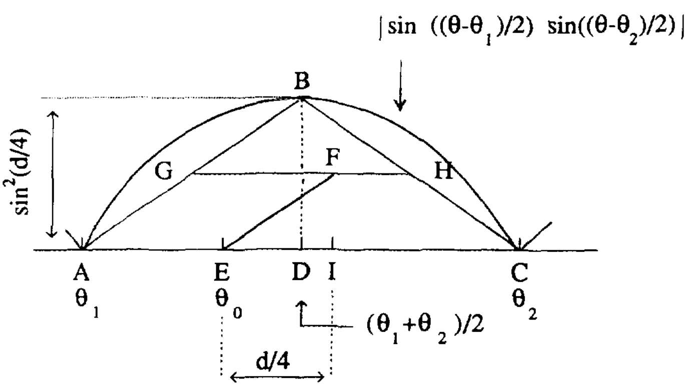

Figure 1

LEMMA 2. *Let* $\theta_1, \theta_2 \in \mathbb{K}$ *satisfying* $d = d_K(\theta_1, \theta_2) > 0$ *and* $g(\theta) = \sin((\theta - \theta_1)/2)\sin((\theta - \theta_2)/2)$. *For any given* $p > 0$ *and* $\alpha > 0$ *there exists a positive constant* $C = C(p, \alpha, d)$ *such that for any* $T \in H_n$

$$\|T\|_p^* \leqslant C n^\alpha \, \|T\, |g|^\alpha\|_p^*.$$

Proof. It is not restrictive to assume that $n \geqslant 2/d$. Let $T \in H_n$ $(n \geqslant 2)$, $\theta_0 \in \mathbb{K}$ be such that $\|T\|_\infty^* = |T(\theta_0)|$ and $J_0 = [\theta_0 - 1/2n, \theta_0 + 1/2n]$. If $\theta \in J_0$ we have $T(\theta) = T(\theta_0) + (\theta - \theta_0)\, T'(\theta')$ for some $\theta' \in J_0$. Then, since $|T'(\theta')| \leqslant n\, \|T\|_\infty^*$ and $|\theta - \theta_0| \leqslant 1/(2n)$,

$$|T(\theta)| \geqslant \tfrac{1}{2}\, \|T\|_\infty^* \; (\theta \in J_0),$$

whence

$$\begin{aligned} &C(p, \alpha)\, n^{-p\alpha - 1}\, \|T\|_\infty^{*p} \\ &\leqslant (1/2)^p\, \|T\|_\infty^{*p}\, (d/\pi^2)^{p\alpha} \int_{\theta_0}^{\theta_0 + 1/2n} |\theta - \theta_0|^{p\alpha}\, d\theta \end{aligned}$$

(we can use Lemma 1 since $1/(2n) < d/4$)

$$\begin{aligned} &\leqslant (1/2)^p\, \|T\|_\infty^{*p} \int_{J_0} |g(\theta)|^{p\alpha}\, d\theta \\ &\leqslant (1/2)^p\, \|T\|_\infty^{*p} \int_{\mathbb{K}} |g(\theta)|^{p\alpha}\, d\theta \\ &\leqslant \int_{\mathbb{K}} |T(\theta)|^p\, |g(\theta)|^{p\alpha}\, d\theta. \end{aligned}$$

Therefore

$$(2/n)\, \|T\|_\infty^{*p} \leqslant C_1(p, \alpha)\, n^{p\alpha}\, \|T\, |g|^\alpha\|_p^{*p}. \tag{4}$$

Let $J_1 = [\theta_1 - 1/n, \theta_1 + 1/n]$, $J_2 = [\theta_2 - 1/n, \theta_2 + 1/n]$. $J_{1,2} = J_1 \cup J_2$, and $J_3 = \mathbb{K} \setminus J_{1,2}$ (since $2/n < d$ we have $J_1 \cap J_2 = \varnothing$). Now, using (4), we get

$$\int_{J_i} |T(\theta)|^p\, d\theta \leqslant (2/n)\, \|T\|_\infty^{*p} \leqslant C_1(p, \alpha)\, n^{p\alpha}\, \|T\, |g|^\alpha\|_p^{*p} \qquad (\mathrm{i} = 1, 2) \tag{5}$$

and due to the fact that for $\theta \in J_3$

$$|g(\theta)| \geqslant \sin(1/(2n))\sin(d/4) \geqslant C(d)/n,$$

we get

$$\int_{J_3} |T(\theta)|^p \, d\theta \leqslant C(d) \, n^{p\alpha} \, \|T \, |g|^\alpha\|_p^{*p}. \tag{6}$$

Inequalities (5) and (6) together give the required result.

LEMMA 3. *Let* $p>0$ *and* $\alpha>0$; *there exists a positive constant* $C=C(p, \alpha)$ *such that, for any* $x_1 \in \mathbb{R}$ *and any* $P \in \mathscr{P}_n$,

$$\int_{-1/2}^{1/2} |P(x)|^p \, dx \leqslant Cn^{p\alpha} \int_{-1}^{1} |P(x)|^p \, |x-x_1|^{p\alpha} \, dx,$$

$$\int_{-1}^{1} |P(x)|^p \, dx \leqslant Cn^{2p\alpha} \int_{-1}^{1} |P(x)|^p \, |x-x_1|^{p\alpha} \, dx.$$

Proof. This is an immediate consequence of [5, Corollary 15, p. 114, Corollary 26, p. 126; 1].

LEMMA 4. *Let* f *be a function such that* $|f'(x)| \geqslant m > 0$ *for* $x \in [-1, 1]$. *Then for any* $p>0$ *and* $\alpha>0$ *there exists a positive constant* $C=C(p, \alpha, m)$ *such that, for any* $P \in \mathscr{P}_n$

$$\int_{-1/2}^{1/2} |P(x)|^p \, dx \leqslant Cn^{p\alpha} \int_{-1}^{1} |P(x)|^p \, |f(x)|^{p\alpha} \, dx,$$

$$\int_{-1}^{1} |P(x)|^p \, dx \leqslant Cn^{2p\alpha} \int_{-1}^{1} |P(x)|^p \, |f(x)|^{p\alpha} \, dx.$$

Proof. Let $x_1 \in [-1, 1]$ be such that $|f(x_1)| = \mathrm{Min}_{|x|<1} |f(x)|$; then $|f(x)| \geqslant |x-x_1| \, m$, and applying Lemma 3 gives immediately the result.

3.2. *Proof of the Theorem in the Particular Case*

Let $P(x, y) = P \in \mathscr{P}_n$.

1. We set

$$Q(r, \theta) = P(r\cos\theta, r\sin\theta), \qquad v(r, \theta) = w(r\cos\theta, r\sin\theta).$$

The assumption that Γ cuts transversally $\partial\Omega$ at only two distinct points A and B implies that $(\partial v/\partial\theta)(A) \neq 0$ and $(\partial v/\partial\theta)(B) \neq 0$. There exists a neighborhood V_A of A and a neigborhood V_B of B such that $V_A \cap V_B = \varnothing$ and $(\partial v/\partial\theta)(x, y) \neq 0$ for $(x, y) \in V_A \cup V_B$. Then one can find $\rho \in (0, r)$ and $d>0$ such that any circle with center at origin and radius $r \in [R-\rho, R]$ cuts Γ (transversally) at only two points whose polar coordinates are

$(r, \theta_A(r))$ and $(r, \theta_B(r))$ with $r \in [R-\rho, R]$ and $d_K(\theta_A(r), \theta_B(r)) \geqslant d > 0$. Thus we can write

$$v(r, \theta) = \sin((\theta - \theta_A(r))/2) \sin((\theta - \theta_B(r))/2)\, u(r, \theta)$$
$$(r \in [R-\rho, R], \theta \in \mathbb{K}),$$

where u is a continuous non-vanishing function satisfying $|u(r, \theta)| \geqslant m > 0$ $(r \in [R-\rho, R],\ \theta \in \mathbb{K})$ for some positive constant m. Using now Lemma 2, we get

$$\int_0^{2\pi} |Q(r, \theta)|^p \, d\theta \leqslant C_1 n^{p\alpha} \int_0^{2\pi} |Q(r, \theta)|^p \, |v(r, \theta)|^{p\alpha} \, d\theta \qquad (r \in [R-\rho, R]).$$

Then, multiplicating both sides by r and integrating with respect to r from $R-\rho$ to R yields

$$\|P\|_{p,E} \leqslant C_1 n^\alpha \, \|P\, |w|^\alpha\|_{p,E}, \tag{7}$$

where E is the annulus $\{(r, \theta);\ \theta \in \mathbb{K},\ r \in [R-\rho, R]\}$.

2. Given a point $S \in \Gamma \cap \Omega$ we can make a change of variable (translation + rotation) such that S becomes the origin and (since $\operatorname{grad} w \neq 0$) $(\partial w/\partial x)(0) \neq 0$. Then we choose $l > 0$ such that

(1) $\mathscr{A} := \{(x, y);\ |x| < l,\ |y| < l\} \subset \Omega$,

(2) for any $(x, y) \in \mathscr{A}$, $|(\partial w/\partial x)(x, y)| \geqslant \frac{1}{2} |(\partial w/\partial x)(0, 0)|$.

Let $\mathscr{O} := \{(x, y);\ |x| < l/2,\ |y| < l\}$ and f be the function defined for a given y $(|y| < l)$ by $f(x) = w(x, y)$. We have $|f'(x)| \geqslant \frac{1}{2} |(\partial w/\partial x)(0, 0)|$ $(|x| < l)$. Then, for any $y \in [-l, l]$ using Lemma 4 gives

$$\int_{-l}^{l} |P(x, y)|^p \, |w(x, y)|^{p\alpha} \, dx \geqslant C_2 n^{-p\alpha} \int_{-l/2}^{l/2} |P(x, y)|^p \, dx.$$

Integrating both sides with respect to y from $-l$ to l yields

$$\|P\|_{p,\mathscr{O}} \leqslant C_3 n^\alpha \, \|P\, |w|^\alpha\|_{p,\mathscr{A}}. \tag{8}$$

3. Let $S \in \Omega \setminus \Gamma$. There exists a neighborhood $\mathscr{B}$ of S such that $\mathscr{B} \subset \Omega$ and $w(x, y) \neq 0$ $((x, y) \in \mathscr{B})$. Then for some positive constant C_4

$$\|P\|_{p,\mathscr{B}} \leqslant C_4 \, \|P\, |w|^\alpha\|_{p,\mathscr{B}} \leqslant C_4 n^\alpha \, \|P\, |w|^\alpha\|_{p,\mathscr{B}}. \tag{9}$$

By a compactness argument, we conclude from (7), (8), and (9) that

$$\|P\|_{p,\Omega} \leqslant C n^\alpha \, \|P\, |w|^\alpha\|_{p,\Omega}. \tag{10}$$

4. Proof of the Theorem in the General Case

1. Let $A \in \Gamma \cap \Omega$; we have $\operatorname{grad} w(A) \neq 0$. Thus, once can find an open disc D with center at A and choose its radius in order that $D \subset \Omega$ and Γ cuts transversally $\partial\Omega$ at only two distinct points. If we use the estimate (10) we see that there exists $C_1 = C_1(D)$ such that for any $P \in \mathscr{P}_n$

$$\|P\|_{p,D} \leqslant C_1 n^\alpha \, \|P\,|w|^\alpha\|_{p,D}.$$

2. Let $A \in \Gamma \cap \partial\Omega$. The curve Γ cuts transversally $\partial\Omega$ at A. Since $\partial\Omega$ is C^2 in a neighborhood of A we can choose a circle $\mathscr{C}$

- whose interior D is included in Ω,
- tangential to $\partial\Omega$ at A,
- with radius r and center C_0, r being small enough so that Γ cuts transversally the circle $\mathscr{C}$ at only two points.

Again, estimate (10) implies that for any $P \in \mathscr{P}_n$

$$\|P\|_{p,D} \leqslant C_2 n^\alpha \, \|P\,|w|^\alpha\|_{p,D}. \tag{11}$$

Let $D(A, n)$ be the disc with center at C_0 and radius $r(1 + 1/n^2)$. By the well-known Bernstein–Walsh inequality [6, Lemma 3]

$$\|P\|_{p,[-1-n^{-2}, 1+n^{-2}]} \leqslant C_3 \, \|P\|_{p,[-1,1]}$$

(where C_3 does not depend on n) is easily extended to the two-dimensional case

$$\|P\|_{p,D(A,n)} \leqslant C_4 \, \|P\|_{p,D}$$

and using (11) we have

$$\|P\|_{p,D(A,n)} \leqslant C_5 n^\alpha \, \|P\,|w|^\alpha\|_{p,D}. \tag{12}$$

3. We remark (this is the crucial point) that $\mathscr{C}$ being tangential to $\partial\Omega$, $\operatorname{dist}(A, \Omega \setminus D(A, n)) \geqslant C_6/n$ for some positive constant C_6 and then one can find N_0 and a finite covering of $\Gamma \cap \bar{\Omega}$ by open discs $D_1, ..., D_k$, $D(A_1, n)$, $D(A_2, n)$, ..., $D(A_r, n)$ whose union V is such that for any $n > N_0$ and any $S \in \Omega \setminus V$, $\operatorname{dist}(S, \Gamma) > C_7/n$ for some positive constant C_7 not depending on n. Furthermore, estimates (11) and (12) yield

$$\|P\|_{p,V} \leqslant C_8 n^\alpha \, \|P\,|w|^\alpha\|_{p,V}. \tag{13}$$

Now, since $\operatorname{grad} w(x, y) \neq 0$ ($(x, y) \in \Gamma$) there exists a positive constant C_9 such that for any $(x, y) \in \Omega$,

$$|w(x, y)| \geqslant C_9 \operatorname{dist}((x, y), \Gamma).$$

Then if $(x, y) \in \Omega \setminus V$ we have $|w(x, y)| \geqslant C_{10}/n$ and

$$\|P\|_{p, \Omega \setminus V} \leqslant C_{11} n^{\alpha} \|P |w|^{\alpha}\|_{p, \Omega \setminus V}. \tag{14}$$

Estimates (13) and (14) together give

$$\|P\|_{p, \Omega} \leqslant C n^{\alpha} \|P |w|^{\alpha}\|_{p, \Omega}.$$

5. Sharpness of the Exponent

5.1. *Some Lemmas*

We need some preliminary estimates on Jacobi polynomials (see [8] for notation). The Jacobi polynomial $P_n^{(d,0)}$ (whose degree is n) satisfies [8, pp. 168–169]

$$|P_n^{(d,0)}(\cos \theta)| \leqslant n^d \quad \text{if } 0 \leqslant \theta \leqslant 1/n \tag{15}$$

$$C n^{-1/2} \theta^{-d-1/2} \quad \text{if } 1/n \leqslant \theta \leqslant \pi/2 \tag{16}$$

$$1 \quad \text{if } \pi/2 \leqslant \theta \leqslant \pi. \tag{17}$$

LEMMA 5. *For $p > 0$, $\alpha \geqslant 0$, and $d > \alpha + 1/p$, there exists a positive constant $C = C(p, \alpha)$ such that for any n*

$$\|P_n^{(d,0)}(1 - 2x^2) |x|^{\alpha}\|_{p, [-1, 1]} \leqslant C n^{d - \alpha - 1/p}. \tag{18}$$

Proof. We have

$$\int_{-1}^{1} |P_n^{(d,0)}(1 - 2x^2)|^p |x|^{p\alpha} \, dx$$

$$= 2 \int_0^1 |P_n^{(d,0)}(1 - 2x^2)|^p |x|^{p\alpha} \, dx$$

$$= \int_0^{\pi} |P_n^{(d,0)}(\cos \theta)|^p |\sin(\theta/2)|^{p\alpha} |\cos(\theta/2)| \, d\theta.$$

Using (15), (16), and (17) we obtain respectively

$$\int_0^{1/n} |P_n^{(d,0)}(\cos \theta)|^p |\sin(\theta/2)|^{p\alpha} |\cos(\theta/2)| \, d\theta$$

$$\leqslant \int_0^{1/n} n^{dp} (\theta/2)^{p\alpha} \, d\theta = C_1 n^{dp - p\alpha - 1},$$

$$\int_{1/n}^{\pi/2} |P_n^{(d,0)}(\cos\theta)|^p\,|\sin(\theta/2)|^{p\alpha}\,|\cos(\theta/2)|\,d\theta$$

$$\leqslant C\int_{1/n}^{\pi/2} (n^{-1/2}\theta^{-d-1/2})^p\,\theta^{p\alpha}\,d\theta \leqslant C_2 n^{dp-p\alpha-1},$$

$$\int_{\pi/2}^{\pi} |P_n^{(d,0)}(\cos\theta)|^p\,|\sin(\theta/2)|^{p\alpha}\,|\cos(\theta/2)|\,d\theta \leqslant C_3.$$

These three estimates together give the required result.

LEMMA 6. *Let $p>0$ and $d>0$. There exists a positive constant $C=C(p,d)$ such that for any $n>0$ we have*

$$\int_{-1/4}^{1/4n} |P_n^{(d,0)}(1-2x^2)|^p\,dx \geqslant Cn^{dp-1}.$$

Proof. Let $Q(x)=P_n^{(d,0)}(1-2x^2)$ and $I=[-1,1]$. We have

$$\|Q\|_{\infty,I}=Q(0)=P_n^{(d,0)}(1)=n^d.$$

By the Markov–Bernstein inequality, for any $x\in I$,

$$|Q'(x)|\leqslant 2n\,\|Q\|_{\infty,I}\,(1-x^2)^{-1/2}.$$

Then for $|x|\leqslant 1/(4n)$

$$|Q(x)-\|Q\|_{\infty,I}|=|Q(x)-Q(0)|=|x|\,Q'(c)$$

for some c between 0 and x. Thus

$$|Q(x)-\|Q\|_{\infty,I}|\leqslant\frac{1}{4n}\,2n\left(1-\frac{1}{16}\right)^{-1/2}\|Q\|_{\infty,I}\leqslant\frac{2}{3}\,\|Q\|_{\infty,I}$$

and then for $|x|\leqslant 1/(4n)$, $|Q(x)|\geqslant\frac{1}{3}\,\|Q\|_{\infty,I}=\frac{1}{3}n^d$; therefore

$$\int_{-1/4n}^{1/4n} |Q(x)|^p\,dx\geqslant Cn^{dp-1}.$$

COROLLARY 1. *Let $p>0$, $d>1/p$. There exists a positive constant $C=C(p)$ such that for any n we have*

$$\int_I |P_n^{(d,0)}(1-2x^2)|^p\,dx\leqslant C\int_{-1/4n}^{1/4n} |P_n^{(d,0)}(1-2x^2)|^p\,dx.$$

Proof. Using (18) with $\alpha=0$ we obtain

$$\int_I |P_n^{(d,0)}(1-2x^2)|^p\,dx \leqslant Cn^{dp-1}.$$

Then Corollary 1 is an immediate consequence of Lemma 6.

5.2. *Proof of the Sharpness*

Let $a \in \Omega \cap \Gamma$. We can assume that a is the origin and that $\Omega \subset [-1,1]^2 = I^2$. Let N be such that $[-1/N, 1/N]^2 \subset \Omega$.

In order to prove the sharpness of the exponent α, we exhibit a sequence $(R_{4n})_{n\in\mathbb{N}}$ of polynomials of degree $4n$ such that

$$n^\alpha \,\| R_{4n}\, |w(x,y)|^\alpha \|_{p,\Omega} \leqslant C\, \|R_{4n}\|_{p,\Omega}.$$

By Taylor's formula, $w(x,y) = w(0,0) + xw'_x(\lambda x, \lambda y) + yw'_y(\lambda x, \lambda y)$ for some $\lambda \in [0,1]$. Since $w(0,0)=0$ we have

$$|w(x,y)|^{\alpha p} \leqslant C_1(|x|^{\alpha p} + |y|^{\alpha p}).$$

We set $R_{4n}(x,y) = Q(x)\,Q(y)$, where $Q(x) = P_n^{(d,0)}(1-2x^2)$ with $d > \alpha + 1/p$.

We have

$$\begin{aligned}
&\int_\Omega |R_{4n}(x,y)|^p\, |w(x,y)|^{\alpha p}\, dx\, dy \\
&\quad\leqslant C_1 \int_\Omega |x|^{\alpha p}\, |Q(x)\,Q(y)|^p\, dx\, dy \\
&\qquad + C_1 \int_\Omega |y|^{\alpha p}\, |Q(x)\,Q(y)|^p\, dx\, dy \\
&\quad\leqslant 2C_1 \int_I |x|^{\alpha p}\, |Q(x)|^p\, dx \int_I |Q(y)|^p\, dy \\
&\quad\leqslant C_2 n^{dp - p\alpha - 1 + dp - 1} \qquad \text{(by Lemma 5)} \\
&\quad\leqslant C_3 n^{-p\alpha} \int_{-1/4n}^{1/4n} |P_n^{(d,0)}(1-2x^2)|^p\, dx \int_{-1/4n}^{1/4n} |P_n^{(d,0)}(1-2y^2)|^p\, dy
\end{aligned}$$

(by Lemma 6 and Corollary 1) and since $[-1/4N, 1/4N]^2 \subset \Omega$

$$\int_\Omega |R_{4n}(x,y)|^p\, |w(x,y)|^{\alpha p}\, dx\, dy \leqslant Cn^{-p\alpha} \int_\Omega |R_{4n}(x,y)|^p\, dx\, dy,$$

which completes the proof.

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